Ramanujan and the Julia Set of the Iterated Exponential Map

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The Julia set of Ramanujan's iterated exponential map, $F_{r+1}(z) = \exp\{F_r(z)\} - 1$, is investigated for positive integral r. Connection with Devaney's work on the map $z_{r+1} = \lambda \exp[z_r]$ is also made.

In his first notebook [1], Ramanujan discussed the ordinary composition of functions and extended that concept to the *fractional order of functions*. In the process, he defined the iterative exponential mapping

$$F_{r+1}(x) = \exp\{F_r(x)\} - 1, \tag{1}$$

where r could be any real number, negative or positive, and x lay on the real line. It would be usual for Ramanujan to have $F_0(x) = x$ as the identity function. Iterated exponential functions have since been extensively studied, and Chapt. 4 of Berndt's book [1] cites an impressive bibliography. The mapping (1) has several remarkable properties, one of which is

$$\partial F_r = [1 + F_r]\, \partial F_{r-1}, \tag{2a}$$

which leads to

$$\partial F_r = \prod_{k=1,2,\ldots r} [1 + F_k], \tag{2b}$$

in which $\partial F \equiv \mathrm{d}F/\mathrm{d}x$. Furthermore,

$$F_{k+r} = F_r(F_k) = F_k(F_r). \tag{2c}$$

It is to be noted that the iterative process (1) can be extended to complex numbers $z = x + i\,y$, because the mapping (1) is holomorphic. In particular, if interest is confined to integral $r \geq 0$, the Julia set for the exponential map can be obtained, and the computer program given by Lakhtakia et al. [2] was adapted for this purpose. Double precision complex arithmetic was used while implementing the program on a DEC VAX 11/730 minicomputer.

Shown in Figs. 1–5 are several portions of the Julia set obtained for the process

$$F_{r+1}(z) = \exp\{F_r(rz)\} - 1; \quad F_0(z) = z. \tag{4}$$

The black portions of these figures refer to those z for which the process (4) converges; while for the z in the white areas, the process shows clearly diverging trends. The boundary between the black and white regions is the Julia set. It is very clear from these figures that this process has a fixed point at $z = 0$, because $F_r(0) \equiv 0\, \forall\, r$. On the real line $y = 0$, $F_1(z) = \exp[x] - 1$. For $x \neq 0$, it is well known that $\exp\{x\} - 1 > x$ [3]. Therefore, as r increases $F_r(x)$ decreases (resp. increases) for $x < 0$ (resp. $x > 0$). On the imaginary line $x = 0$, $F_1(z) = \exp[i\,y] - 1 = (\cos y - 1) + i \sin y$, which is always bounded. Hence, the iterative process always converges when $x = 0$.

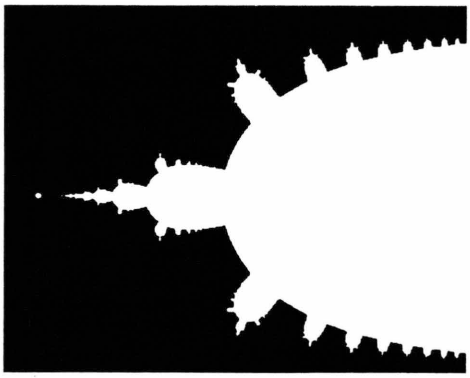

Fig. 1. Portion of the Julia set of the iterated exponential map (4). The argument $z = x + i\,y$, $-4 \leq x \leq 4$, $-3.046 \leq y \leq 3.046$. 32 iterations were used for the calculations of Fig. 1–5 using complex double precision arithmetic.

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The real line $y = 0$ is also the only symmetry axis of this Julia set. This is not surprising if both $F_r(z)$ and $F_{r+1}(z)$ are expanded as Taylor series:

$$F_r(z) = \sum_{k=0,1,2,3,\ldots} (\alpha_k/k!)\, z^k; \tag{5a}$$

$$F_{r+1}(z) = \sum_{k=0,1,2,3,\ldots} (\beta_k/k!)\, z^k. \tag{5b}$$

By substituting (5a, b) in (4) and equating the like powers of z, it turns out that

$$\beta_k = \sum_{j=0,1,2,\ldots,k} ({}_{k-j}j)\,\alpha_j, \tag{6}$$

in which $({}_{k-j}j)$ is the Stirling number of the second kind [4]. On observing that $F_0(z) = z$ and $F_1(z) = z + z^2/2! + z^3/3! + \ldots$, it is clear that the lowest exponent of z in the polynomial expansion of F_r is unity for all r considered. As per Lakhtakia et al. [2], therefore, this Julia set can possess only one axis of symmetry.

To be noted is that Figs. 1 and 2 clearly bring out the fact that the Julia set contains a cascaded arrangement of a unit feature, the size of the unit feature increasing as one proceeds alongs the $+x$ axis, and the size of the unit feature generally decreasing as one moves away from the x axis in either direction. This suggests that the Julia set has a self-affine character. From observing Fig. 1–4, it is clear that the Julia set

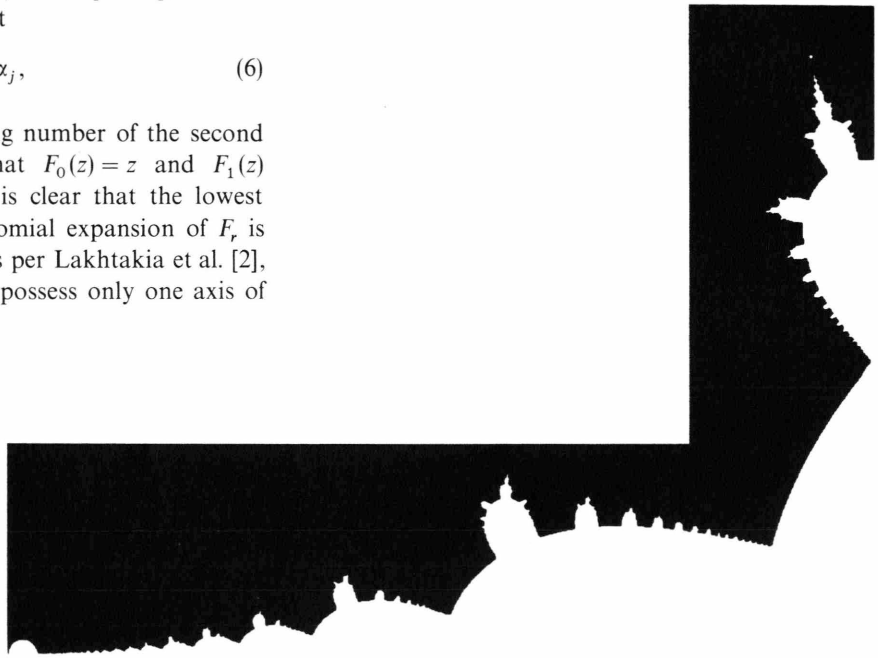

Fig. 2. Same as Fig. 1 but $0 \le x \le 2$ and $0 \le y \le 1.523$.

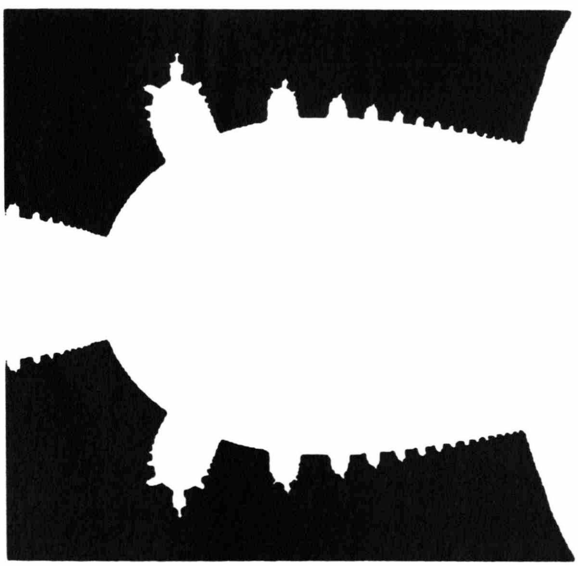

Fig. 3. Same as Fig. 1 but $0.8435897 \le x \le 2.156411$ and $-0.5 \le y \le 0.5$.

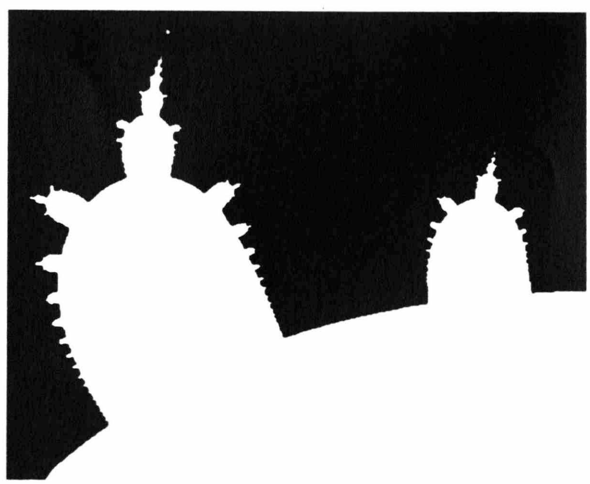

Fig. 4. Same as Fig. 1 but $1.0 \le x \le 1.3938463$ and $0.2 \le y \le 0.5$.

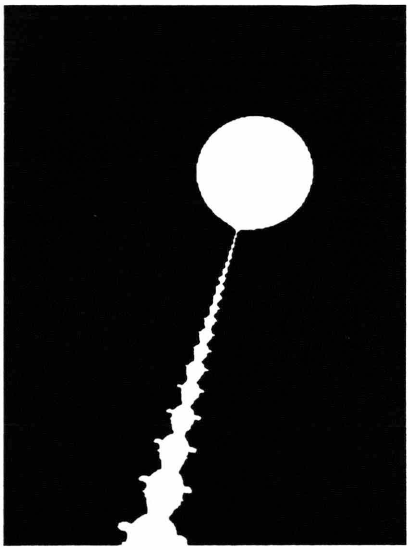

Fig. 5. Same as Fig. 1 but $1.145 \leq x \leq 1.175$ and $0.4467 \leq y \leq 0.4696$.

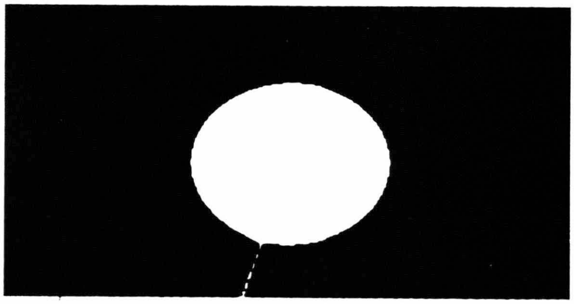

Fig. 6. Magnified view of the circular termination of Fig. 5; $1.154 \leq x \leq 1.161$ and $0.4581 \leq y \leq 0.4610$, and 64 iterations were used for the calculations using complex double precision arithmetic.

is made up of smooth curves of appreciable length which join to form cusps. Even more curiously, as exemplified by Fig. 5, at its every extremity the Julia set has a circular segment. The circular termination is certainly curious: in Fig. 6 we zoom onto it and show, by using a different number of iterations, that it is not a numerical artifact. These two observations suggest that the length of some portions of the Julia set may not increase ad infinitum under microscopic examination; it cannot be denied, however, that these characteristics may be due to finite precision arithmetic, and a more intensive analytic examination is desired.

The interior of the Julia set forms the basin of attraction. For the polynominal maps of Lakhtakia et al. [2], the basins of attraction are finite in area. As can be inferred from Fig. 1, however, the black area, which forms the basin of attraction, is obviously not so. This observation finds corroboration by Cvitanovic et al. [5].

The mapping (4) is simply a special case of the general family

$$z_{r+1} = \lambda \exp[z_r] \tag{7}$$

studied by Devaney [6]. It was shown by Misiurewicz [7] that the Julia set for the process (7) is the whole complex plane itself when $\lambda = 1$. This conclusion has been further shown to be valid by Devaney for $\lambda > 1/e$. On the other hand, for $0 < \lambda < 1/e$, the Julia set has been shown to be a Cantor set of curves, called hairs by Devaney, as exemplified by Fig. 2 of Devaney's paper. To quote Peitgen and Richter [8], "as λ increases and passes through $1/e$," the Julia set "experiences an explosion. Pictures of Julia sets for this family are quite difficult to obtain..."

As noted above, when $\lambda > 1/e$, then the Julia set of (7) is the complex plane itself. For $0 < \lambda < 1/e$, the Cantor set-like character of the Julia set has been illustrated in Devaney's paper. The mapping (4) coincides with the mapping (7) when $\lambda = 1/e$; hence, Fig. 1–6 give the Julia set for the mapping $z_{r+1} = \exp[z_r]/e$. From these figures it is evident that hair-like structures may evolve if λ decreases below $1/e$. At $\lambda = 1/e$, however, the circular terminations, especially in Figs. 5 and 6, exemplify the hairs being primed for "explosion" as λ increases beyond $1/e$.

In 1987 the 100-th anniversary of the birth of Ramanujan was celebrated. In closing, we wonder if that untutored genius had ever thought of fractals and chaos.

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